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LETTER TO THE EDITOR

The random Fibonacci recurrence and the visible points of the plane

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Online at stacks.iop.org/JPhysA/39/L323**Abstract**

In this letter, we show a connection between the random Fibonacci recurrence and the visible points of the plane. In particular, we show that by suitably modifying the rules of the random Fibonacci map, there is a unique correspondence between the visible points (points with relative prime coordinates) of the first quadrant and the vertices of a self-similar graph (what we call the Fibonacci graph). The proposed random recurrence can then be interpreted as a random walk on this graph.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

A variant of the well-known Fibonacci series is the so-called random Fibonacci recurrence $x_{n+1} = x_n \pm x_{n-1}$ (Viswanath 2000). This recurrence can also be written as the second-order difference equation (two-dimensional map)

$$\begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix} = D_n \begin{pmatrix} x_{n-2} \\ x_{n-1} \end{pmatrix}, \quad (1)$$

where the coefficient matrix D_n at time n is randomly (with probability 1/2) chosen from the set $\{A, B\}$, with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}. \quad (2)$$

Viswanath (2000) showed that $|x_n|$ diverges (almost surely) as γ^n , where $\gamma = \lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = 1.131\,988\dots$ using the theory of random matrix products (Furstenberg's theorem), the Stern–Brocot division of the real line and a computer-assisted integration over a fractal measure. Embree and Trefethen (1999) numerically investigated $x_{n+1} = x_n \pm \beta x_{n-1}$, a generalization

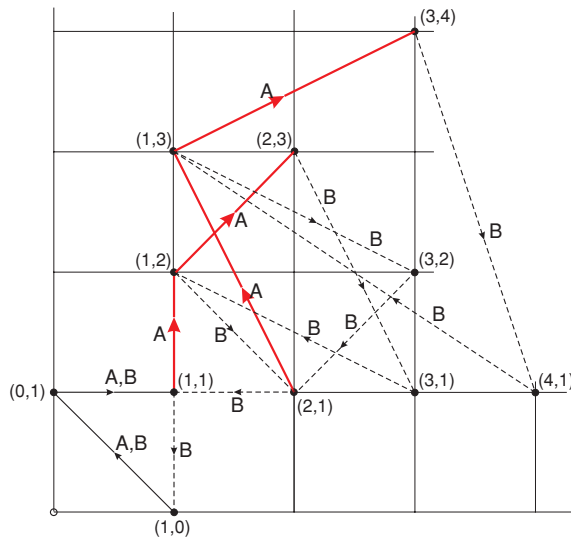


Figure 2. The random Fibonacci maps on the first quadrant.

absolute values of the sequence elements, we exploit the apparent symmetry of this structure to define ‘equivalent’ maps acting on the first quadrant of the integer lattice \mathbb{Z}^2 . In particular, we note that the four points $(i, j), (-i, j), (i, -j), (-i, -j), i, j \in \mathbb{Z}^+$, get mapped to $(j, i+j), (j, i-j), (j, j-i), (j, -i-j), (-j, i-j), (-j, i+j), (-j, -i-j), (-j, j-i)$. These coordinates have absolute value of either $(j, i+j)$ or $(j, |i-j|)$. Thus, we introduce the modified random Fibonacci recurrence with the maps

$$\begin{aligned}
 \mathbf{A} &: (i, j) \rightarrow (j, i+j) \\
 \mathbf{B} &: (i, j) \rightarrow (j, |i-j|) \quad i, j > 0.
 \end{aligned}
 \tag{5}$$

Figure 2 depicts the geometry of the mappings, while figure 3 shows some points resulting from the repeated applications of \mathbf{A} and \mathbf{B} (with $(1, 1)$ as the initial point). Here, we make two observations:

- (i) The coordinates of points in figure 3 are relative primes.
- (ii) Three-link loops can be seen in figure 2, meaning that the application of \mathbf{A} followed by \mathbf{B} twice results in a closed path.

Points with relative prime coordinates are also called visible points, implying that there is no other lattice point on the segment joining them with the origin. The structure in figure 2 can also be interpreted as a directed graph (what we call the Fibonacci graph \mathcal{F}) and in the next section we explore its topology.

3. The Fibonacci graph and the visible points

First we prove observation (2) of the previous section, i.e. that $(\mathbf{B} \circ \mathbf{B} \circ \mathbf{A})(i, j) = (i, j)$. First, \mathbf{B} is replaced with the two linear maps

$$\begin{aligned}
 B_1 &: (i, j) \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} (i, j) = (j, j-i) & i < j \\
 B_2 &: (i, j) \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} (i, j) = (j, i-j) & i > j.
 \end{aligned}
 \tag{6}$$

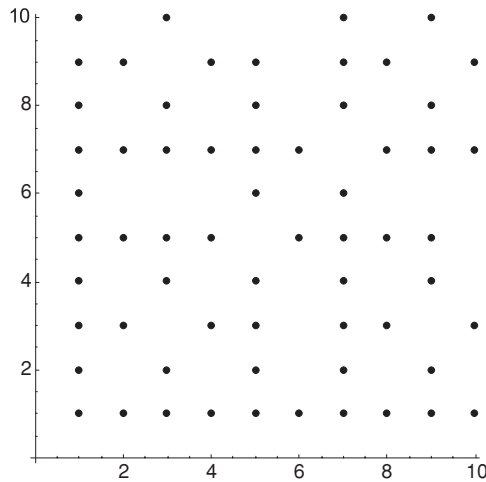


Figure 3. Visible points on the lattice Z^2 .

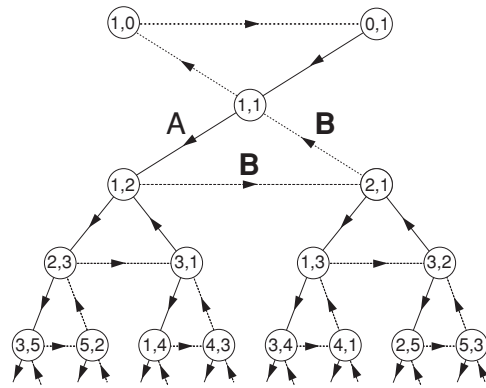


Figure 4. The Fibonacci graph.

We call a lattice point type a if its first coordinate is less than the second (i.e., it is above the diagonal) and type b if the first coordinate is greater than the second (the point is below the diagonal). It is clear that A maps (i, j) to $(j, i + j)$, a type a point. When B is applied to such a point, the new coordinates can be calculated by applying B_1 , resulting in $(i + j, i)$, a point of type b. Since now the second coordinate is smaller, the effect of B on this point is equivalent to that of B_2 , leading back to (i, j) . Indeed,

$$(B \circ B \circ A)(i, j) = B_2 B_1 A(i, j) = (i, j). \tag{7}$$

Using this property of the maps, we can ‘unfold’ the Fibonacci graph as illustrated in figure 4. Clearly, this graph has 2^n distinct nodes (2^{n-1} type a points and 2^{n-1} type b points) at the n th level ($n = 0, 1, \dots$). This structure is similar to several objects studied previously, such as the Stern–Brocot tree, the Calkin–Wilf tree (Calkin and Wilf 2000) and the construction of Alexander and Zagier (1991), that they also called the Fibonacci graph.

Now we return to observation (1). A point with integer coordinates (i, j) is called *visible* iff $\gcd(i, j) = 1$. Mosseri (1992) proved that the geometrical structure associated with the

set of visible points V is the Möbius transform of the original lattice. This set has a D_4 dihedral group symmetry, non-periodic and is invariant to the action of $GL(2, \mathbb{Z})$ (the group of integer 2×2 matrices with determinant ± 1). The invariance property is seen from the fact that visible points are transformed to visible points (Hardy and Wright 1979). Note that the determinants of A, B_1, B_2 are $1, -1, -1$, respectively. Lagarias and Tresser (1995) showed that the rational numbers can be presented as the set of vertices of a degree-three tree (what they call the extended Farey tree). Here, we will show that the rational numbers can also be assigned to the vertices of a connected, self-similar graph (what we call the Fibonacci graph).

We say that a lattice point (k, l) is *reachable* from another lattice point (i, j) , if there exist a sequence of transformations (using A and B) that takes (i, j) to (k, l) . Then (i, j) is reachable from $(1, 1)$ iff (i, j) is visible. Moreover, every visible point is reachable from every visible point. To show this we construct the sequence of transformations taking (i, j) to $(1, 1)$, thereby effectively providing the *address* of a point. The algorithm is simple and can be read off from figure 4. If (i, j) is type a then the trailing letter of the address is A , calculate the new (i, j) as $A^{-1}(i, j) = (j - i, i)$. If (i, j) is type b, then the trailing letter of the address is BA , calculate the new (i, j) as $(BA)^{-1}(i, j) = B_2^{-1}(i, j) = (j, i - j)$. These steps are repeated and the algorithm is terminated at $(i, j) = (1, 1)$. Therefore, the Fibonacci graph is connected and its vertices are the visible points.

The modified random Fibonacci recurrence (5) thus corresponds to a random walk on this structure. The proposed algorithm is essentially the same as the subtractive version of Euclid's algorithm.

4. Discussion

The set of visible points has many interesting properties. For example, the density of the visible points is $1/\zeta(2) = 6/\pi^2$ (Apostol 1976, section 3.8). This is equivalent to saying that the probability of two integers chosen at random being relatively prime is $6/\pi^2 \sim 0.608$. Baake *et al* (1994) calculate the Fourier transform (structure factor) of this set, continuing and generalizing the work by Schroeder (1982) and Mosseri (1992). Due to the correspondence between the modified random Fibonacci recurrence and a random walk on the visible points, properties of this set can be extracted as averages of quantities associated with random walks on graphs (Burioni and Cassi 2005). This formulation might also provide a novel way to evaluate number theoretical products (like the ones in Campbell (1994)). Since visible points are intimately related to the structure of quasicrystals, these developments are expected to further research in that area as well. Another interesting topic worth pursuing is the random walk on self-similar structures. For example, Teufl (2002) investigates asymptotic properties of random walks on self-similar graphs. In a future paper, we will characterize some properties associated with a random walk on the Fibonacci graph as well as on n -loop self-similar graphs.

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